Finite Volume for Fusion Simulations

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Finite Volume for Fusion Simulations

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Plan

1. MHD equations and issues
2. Free-divergence
   - Well-known methods
   - Vector potential
   - Divergence cleaning methods
   - Proposed method
3. Scheme with projection
   - 2-D system
   - Evolution step
   - Scheme with projection
4. Numerical tests
   - Brio-Wu
   - Orszag-Tang
   - Kelvin-Helmholtz instabilities
   - Screw pinch equilibrium

Resistive MHD equations

- Non-conservative equations: $E^{hd} = \frac{p}{\gamma-1} + \frac{1}{2}\rho\mathbf{u}^2$, $\mathbf{E} = \mathbf{B} \times \mathbf{u} + \eta\mathbf{J}$

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) & = & 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p & = & \mathbf{J}\times\mathbf{B}, \\ \partial_t E^{hd} + \nabla\cdot\left[(E^{hd}+p)\mathbf{u}\right] & = & (\mathbf{J}\times\mathbf{B})\cdot\mathbf{u} + \eta\mathbf{J}^2, \\ \partial_t\mathbf{B} + \nabla\times\mathbf{E} & = & 0. \end{cases}$$

- Conservative equations: $E = E^{hd} + \frac{1}{2}\mathbf{B}^2$, $p^* = p + \frac{1}{2}\mathbf{B}^2$

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) & = & 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} - \mathbf{B}\otimes\mathbf{B}) + \nabla p^* & = & 0, \\ \partial_t E + \nabla\cdot[(E+p^*)\mathbf{u} - (\mathbf{u}\cdot\mathbf{B})\mathbf{B}] & = & -\nabla\cdot(\eta\mathbf{J}\times\mathbf{B}), \\ \partial_t\mathbf{B} + \nabla\cdot(\mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u}) & = & -\nabla(\eta\mathbf{J}). \end{cases}$$

MHD system

- Hyperbolic part of the system: 3 wave types
  - 2 Alfvén waves,
  - 4 magneto-acoustic waves (2 fast and 2 slow),
  - 2 material waves (moving with $\mathbf{u}$).
- Involution equation:

$$\partial_t(\nabla \cdot \mathbf{B}) = 0.$$

Then

$$\nabla \cdot \mathbf{B}(t=0) = 0 \Rightarrow \nabla \cdot \mathbf{B}(t) = 0 \ \forall t.$$

- Numerical issues:
  1. Shock capturing: need a robust and accurate scheme.
  2. Free-divergence constraint.
  3. Time scale $T >> T_{alfven} = L/v_{alfven}$: need implicit scheme.

Free-divergence constraint: Well-known methods

2 families of methods:

- Vector potential $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

- Divergence cleaning methods:

$$\text{Enforce } \nabla \cdot \mathbf{B} = 0.$$

Vector potential A

Replace **B** by **A** in the equation:

$$\partial_t \mathbf{A} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \eta \nabla \times (\nabla \times \mathbf{A}) = -\nabla U.$$

Advantage:

- Insure $\nabla \cdot \mathbf{B} = 0$.

Drawbacks:

- One order higher in the spatial derivatives.
- More complex system of equation.

Divergence cleaning methods: Previous works

- Powell et al. (JCP, 1999): Add a source term proportionnal to $\nabla \cdot \mathbf{B}$.

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -(\nabla \cdot \mathbf{B})\mathbf{u}.$$

- Munz et al. (JCP, 2000), Dedner et al. (JCP, 2001): Generalized Lagrange Multiplier (GLM) method.

$$\begin{aligned} \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) + \nabla \Psi &= 0, \\ \mathcal{D}(\Psi) + \nabla \cdot \mathbf{B} &= 0, \\ \mathcal{D}(\Psi) &= \frac{1}{c_h^2} \partial_t \Psi + \frac{1}{c_p^2} \Psi. \end{aligned}$$

- Balsara and Spicer (JCP, 1998): Constrained transport method.

$$\frac{d}{dt} \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathcal{S} = - \oint_{\partial \mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{l}.$$

Divergence cleaning methods

Advantages:

- Easily incorporated with a Riemann type scheme.

Drawbacks:

- Divergence cleaning:
  - $\nabla \cdot \mathbf{B}$ appears in several equations.
  - Not approximated with the same discretization.
  - Divergence cleaning method insure $\nabla \cdot \mathbf{B} = 0$ for one discrete approximation.
- Constrained transport: difficult to implement for arbitrary meshes.

Proposed method

- A mixture of the two methods.
- Idea: work with a redundant system.
- Both vector potential and magnetic fields equations.
- Based on the relaxation scheme method:
  1. Evolution step: FV method for all the variables.
  2. Projection step: enforce $\mathbf{B} = \nabla \times \mathbf{A}$.

2-D implementation

- Scalar potential ψ:

$$\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi.$$

- Augmented system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) & = & 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} - \mathbf{B} \otimes \mathbf{B}) + \nabla p^* & = & 0, \\ \partial_t E + \nabla \cdot [(E + p^*)\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}] & = & -\eta \nabla \cdot (\mathbf{J} \times \mathbf{B}), \\ \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) & = & \eta \nabla^2 \mathbf{B}, \\ \partial_t \psi + \mathbf{u} \cdot \nabla \psi & = & \eta \nabla^2 \psi. \end{cases}$$

Evolution step

- System $\partial_t U + \partial_x F(U) + \partial_y G(U) = 0$:

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{\Delta x}(F_{i+1/2,j} - F_{i-1/2,j}) - \frac{\Delta t}{\Delta y}(G_{i,j+1/2} - G_{i,j-1/2}).$$

- $F_{i+1/2,j}$ and $G_{i,j+1/2}$ computed with a Riemann solver.

Evolution step: HLLD scheme

- Miyoshi and Kusano (JCP, 2005).
- Approximate Riemann solver with 4 intermediate states.

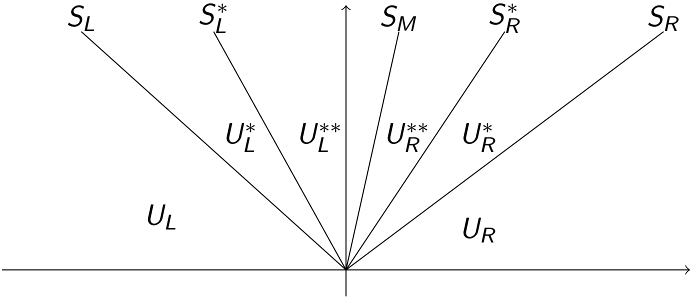

Figure : Riemann fan with four intermediate states.

Evolution step: HLLD scheme

- Intermediate fluxes:

$$\left\{\begin{array}{lcl} F_K^* & = & F_K + S_K U_K^* - S_K U_K \\ F_K^{**} & = & F_K + S_K^* U_K^{**} - (S_K^* - S_K) U_K^* - S_K U_K \end{array}\right., \quad K = L, R.$$

- $\rho\psi$ flux:

$$F(\rho\psi) = F(\rho) \times \left\{\begin{array}{ll} \psi_L, & 0 \leq S_M \\ \psi_R, & 0 \geq S_M \end{array}\right. .$$

Scheme with projection

1. Evolution step: HLLD scheme

$$\left\{ \begin{array}{l} \mathbf{B}_{i,j}^{n+1/2}, \\ \psi_{i,j}^{n+1/2}. \end{array} \right.$$

2. Projection step:

$$\left\{ \begin{array}{lcl} \psi_{i,j}^{n+1} & = & \psi_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1} & = & B_{z,i,j}^{n+1/2}\mathbf{e}_z + \mathbf{e}_z \times (\nabla\psi)_{i,j}^{n+1}. \end{array} \right.$$

Numerical tests

Shock capturing tests:

- Brio-Wu,
- Orszag-Tang,
- Kelvin-Helmholtz.

Plasma fusion tests:

- Screw pinch equilibrium.

1-D Brio-Wu shock tube

- Magnetic field:

$$\mathbf{B} = B_x \mathbf{e}_x + \partial_x \psi \mathbf{e}_y.$$

- 1-D equation for the potential ψ:

$$\partial_t(\rho\psi) + \partial_x(\rho\psi u) = B_x \rho v.$$

- Initial conditions:

	ρ	$\mathbf{u}$	p	B_x	$B_y = \partial_x\psi$	ψ
$x < 0.5$	1	0	1	0.75	1	x
$x > 0.5$	0.125	0	0.1	0.75	-1	$1 - x$

- 1-D mesh: 100 points
- Final time: 0.1

1-D Brio-Wu: Magnetic component B_y

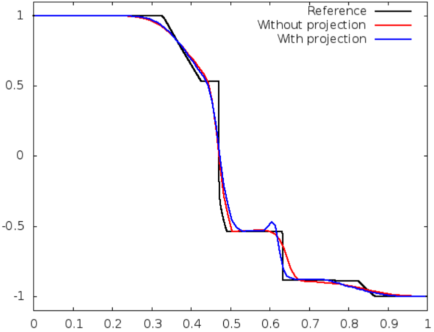

Figure : HLLD O(1).

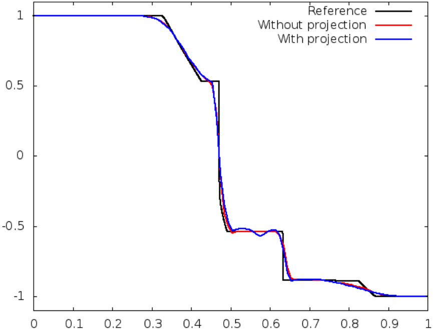

Figure : HLLD O(2).

2-D Brio-Wu shock tube

- Potential:

$$\psi(x,y) = \begin{cases} x - 0.75y, & x < 0.5 \\ 1 - x - 0.75y, & x > 0.5 \end{cases} .$$

- 2-D mesh: 100×10 cells.

2-D Brio-Wu shock tube

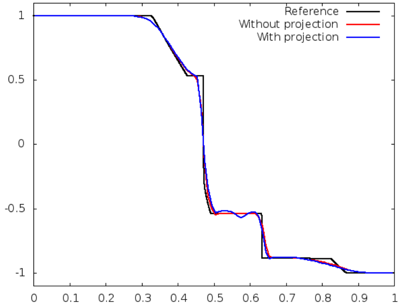

Figure : B_y HLLD O(2).

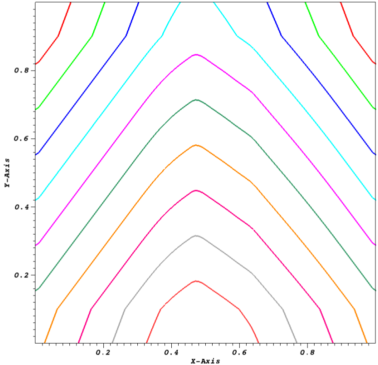

Figure : ψ, HLLD O(2).

Orszag-Tang

- Initial conditions: $\gamma = 5/3$

ρ	$u(x,y)$	$v(x,y)$	$p(x,y)$	$B_x(x,y)$	$B_y(x,y)$	$\psi(x,y)$
γ^2	$-\sin(2\pi y)$	$\sin(2\pi x)$	γ	$-\sin(2\pi y)$	$\sin(4\pi x)$	$-\frac{1}{2\pi}\cos(2\pi y) - \frac{1}{4\pi}\cos(4\pi x)$

- Mesh: 512×512 cells.
- Final time: 0.5.

Orszag-Tang: Pressure field

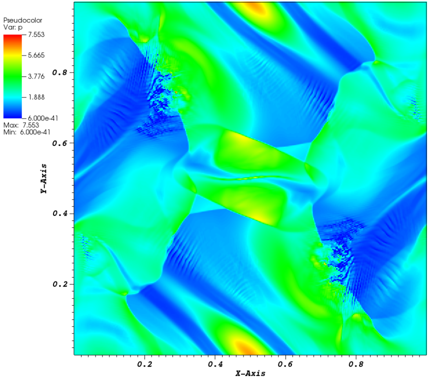

Figure : Without projection.

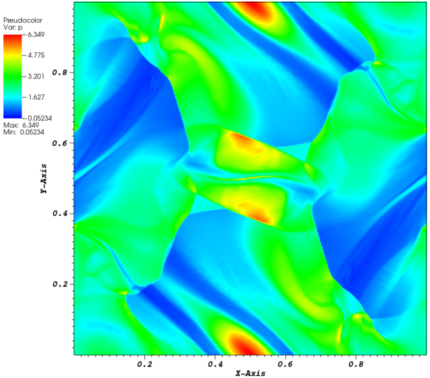

Figure : With projection.

Orszag-Tang

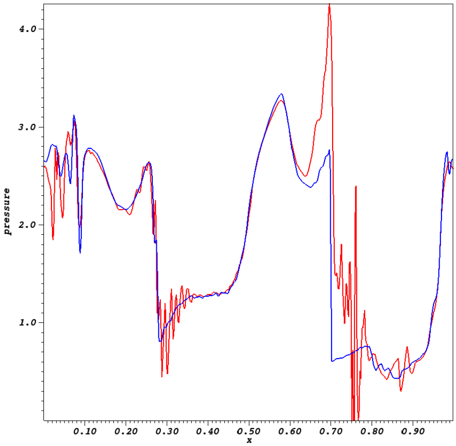

Figure : Pressure at $y = 0.3125$ Red: without projection, Blue: with projection.

Orszag-Tang

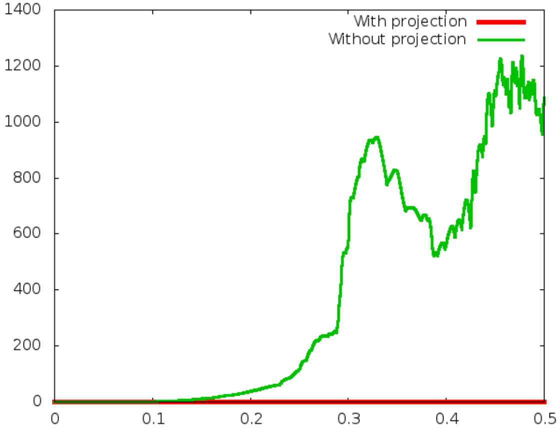

Figure : $\|\nabla \cdot \mathbf{B}\|_\infty$.

Kelvin-Helmholtz instabilities

- Initial data

ρ	p	$u(x,y)$	$v(x,y)$	$B_x(x,y)$	$B_y(x,y)$	$B_z(x,y)$	$\psi(x,y)$
1	$\frac{1}{\gamma}$	$\frac{1}{2}\tanh(\frac{y}{y_0})$	0	$0.1\cos(\frac{\pi}{3})\sqrt{\rho}$	0	$0.1\sin(\frac{\pi}{3})\sqrt{\rho}$	$-0.1\cos(\frac{\pi}{3})\sqrt{\rho}y$

- Single mode pertubation

$$v(x,y) = 0.01\sin(2\pi x)\exp(-\frac{y^2}{\sigma^2}),\ \ \sigma = 0.01.$$

- Mesh: 256×512 points
- $\frac{B_{pol}}{B_{tor}} = \frac{\sqrt{B_x^2+B_y^2}}{B_z}$.

Kelvin-Helmholtz: $t = 5.0$

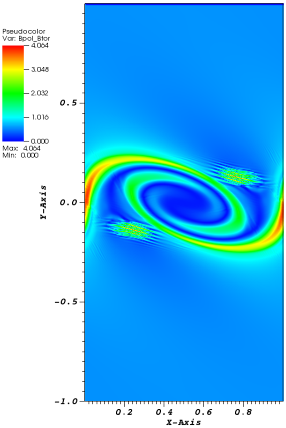

Figure : Without projection.

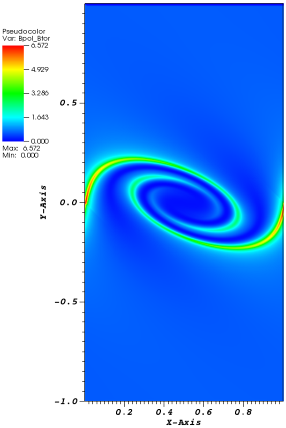

Figure : With projection.

Kelvin-Helmholtz: $t = 8.0$

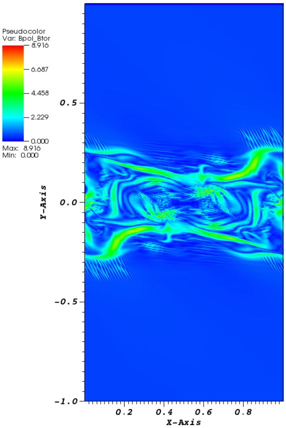

Figure : Without projection.

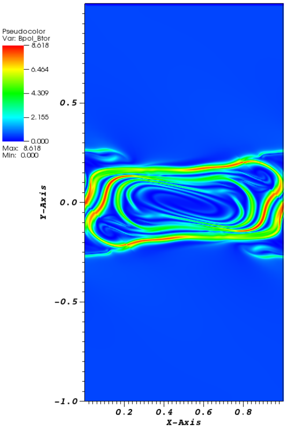

Figure : With projection.

Kelvin-Helmholtz: $t = 12.0$

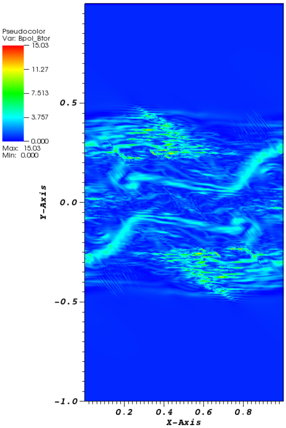

Figure : Without projection.

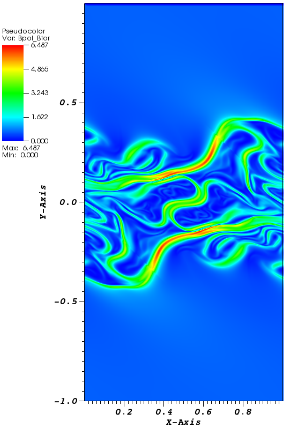

Figure : With projection.

Kelvin-Helmholtz: $t = 20.0$

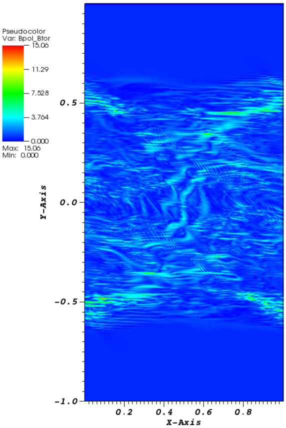

Figure : Without projection.

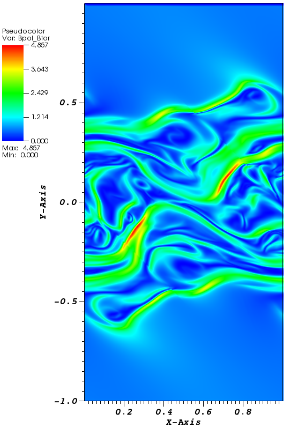

Figure : With projection.

Kelvin-Helmholtz

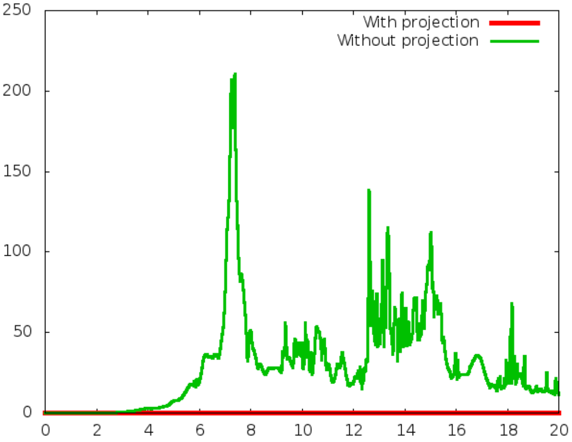

Figure : $||\nabla \cdot \mathbf{B}||_\infty$.

Preliminary test for fusion plasma

- Equilibrium computation: $\nabla p = \mathbf{J} \times \mathbf{B}$.
- Initial conditions:

$$\begin{cases} R_0 = 10, & B_r = 0, \\ \rho = 1, & B_\theta(r) = \frac{r}{R_0(3r^2+1)}, \\ \mathbf{u} = 0, & B_z = 1, \\ p(r) = \frac{1}{6R_0^2(3r^2+1)^2}, & \psi(r) = \frac{1}{6R_0}\ln(3r^2+1). \end{cases}$$

- Aligned mesh: cylindrical coordinates (100×10 cells).
- Non aligned mesh: Cartesian coordinates (200×200 cells).

Cylindrical coordinates

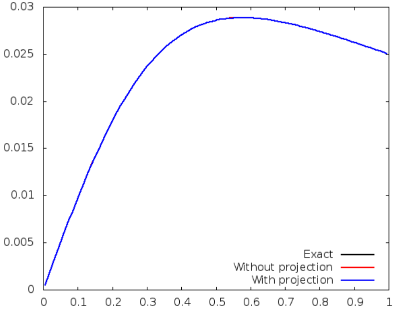

Figure : B_θ.

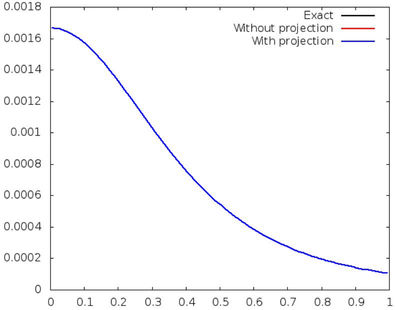

Figure : p.

Cylindrical coordinates: steady state

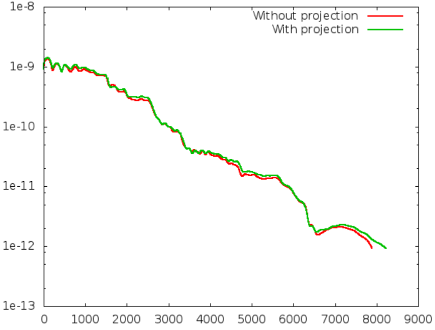

Figure : Residual on r-momentum equation.

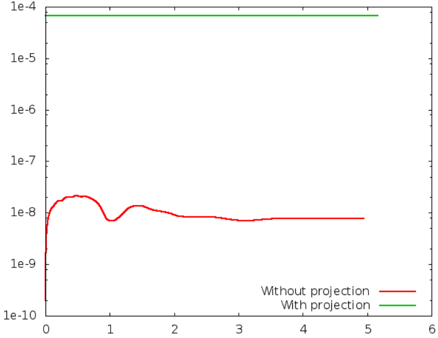

Figure : Relative error of B_θ.

Cartesian coordinates

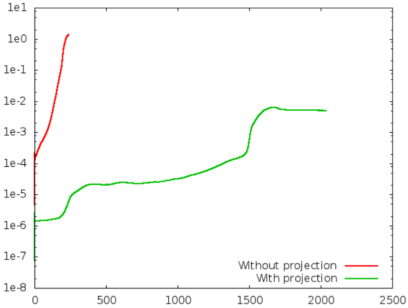

Figure : Relative error of p in function of Alfvén time.

- Non aligned meshes: with projection simulations maintains the equilibrium on 1000 Alfvén times (around 0.5*ms* and 10^6 time steps).
- However, we want to perform the simulation on several milliseconds.

Remedies

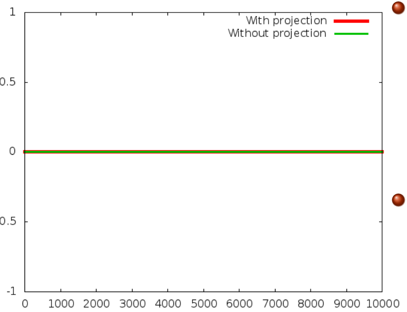

Figure : Relative error of p.

- A simple trick: remove the truncation error evaluated at time $t = 0$.

$$\begin{aligned}\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p & \\ -\mathbf{J} \times \mathbf{B} - (\nabla_h p - \mathbf{J}_h \times \mathbf{B}_h)_{eq} &= 0.\end{aligned}$$

- Well-balanced scheme: work in progress.

Conclusions and perspectives

Conclusions:

- Shock capturing tests: Scheme with projection gives satisfactory results.
- Plasma fusion test: Work in progress on well-balanced scheme.

Perspectives:

- Perform more tests for plasma fusion (kink instability).
- Adapt to 3-D geometry.
- Test on resistive problems.

Cylindrical equations

$$
\left\{
\begin{array}{lcl}
\partial_t(r\rho) + \partial_r(r\rho u_r) + \partial_\theta(\rho u_\theta) & = & 0, \\
\partial_t(r\rho u_r) + \partial_r\left[r(\rho u_r^2 + p^* - B_r^2)\right] + \partial_\theta(\rho u_r u_\theta - B_r B_\theta) & = & \rho u_\theta^2 + p^* - B_\theta^2, \\
\partial_t(r^2\rho u_\theta) + \partial_r\left[r^2(\rho u_r u_\theta - B_r B_\theta)\right] + \partial_\theta\left[r(\rho u_\theta^2 + p^* - B_\theta^2)\right] & = & 0, \\
\partial_t(r\rho u_z) + \partial_r\left[r(\rho u_r u_z - B_r B_z)\right] + \partial_\theta(\rho u_\theta u_z - B_\theta B_z) & = & 0, \\
\partial_t(rE) + \partial_r\left[r[(E + p^*)u_r - (\mathbf{u}\cdot\mathbf{B})B_r]\right] + \partial_\theta\left[(E + p^*)u_\theta - (\mathbf{u}\cdot\mathbf{B})B_\theta\right] & = & 0, \\
\partial_t(rB_r) + \partial_\theta(u_\theta B_r - u_r B_\theta) & = & 0, \\
\partial_t B_\theta + \partial_r(u_r B_\theta - u_\theta B_r) & = & 0, \\
\partial_t(rB_z) + \partial_r\left[r(u_r B_z - u_z B_r)\right] + \partial_\theta(u_\theta B_z - u_z B_\theta) & = & 0, \\
\partial_t(r\rho\psi) + \partial_r(r\rho\psi u_r) + \partial_\theta(\rho\psi u_\theta) & = & 0.
\end{array}
\right.
$$

Scheme with projection

1. Evolution step: HLLD scheme

$$\begin{cases} p_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1/2}, \\ \psi_{i,j}^{n+1/2}. \end{cases}$$

2. Projection step:

$$\begin{cases} \psi_{i,j}^{n+1} & = & \psi_{i,j}^{n+1/2}, \\ \mathbf{B}_{i,j}^{n+1} & = & B_{z,i,j}^{n+1/2}\mathbf{e}_z + \mathbf{e}_z \times (\nabla\psi)_{i,j}^{n+1}, \\ E_{i,j}^{n+1} & = & \frac{p_{i,j}^{n+1/2}}{\gamma-1} + \frac{1}{2}\rho^{n+1/2}\mathbf{u}^2 + \frac{1}{2}(\mathbf{B}_{i,j}^{n+1})^2. \end{cases}$$